A NOTE ON THE IPL AUCTION

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Note: This is a preliminary version. I will update it soon. If you have comments or questions, feel free to email me at soumendu@berkeley.edu.

In this note we will discuss the IPL auction in detail. First, let us recall the mechanism of the IPL auction.

The Indian Premier League (IPL) is a professional cricket league, where different franchises own teams that contest each other. Before the start of each season, an auction is run to distribute players into teams. There is a large pool of players, and each of them is sold to franchises via an auction. For a particular player, there is some base price $\$m$. First an English auction is run. If the bid does not reach $\$m$, the English auction determines the winning franchise. On the other hand, if the bid reaches (or exceeds) $\$m$, then a sealed-bid first price auction is run among the remaining participants, and the highest bidder gets the player at its bid.

Fix a player. Let us assume that there are n franchises with values $V_1, \ldots, V_n \sim F$. Let $(\beta_1^{(1)}, \beta_1^{(2)}), = \cdots = (\beta_n^{(1)}, \beta_n^{(2)}) = (\beta^{(1)}, \beta^{(2)})$ be optimal bidding strategies for the first and the second stages. Then let us compute the allocation probabilities first. Note that there are two contingencies: (a) the first-stage English auction determines the winner, and (b) the second-stage first price auction determines the winner.

Note that (a) occurs iff $\max_k V_k < m$ and (b) occurs iff $\max_k V_k \geqslant m$. Therefore the allocation probabilities are given by

$$\begin{aligned} a_i &= \mathbb{P}(\beta^{(1)}(V_i) > \max_{k\neq i} \beta^{(1)}(V_k), \max_k V_k < m) \\ &\qquad + \mathbb{P}(\beta^{(2)}(V_i) > \max_{k\neq i} \beta^{(2)}(V_k), \max_k V_k \geqslant m) \\ &= \mathbb{P}(\beta^{(1)}(V_i) > \max_{k\neq i} \beta^{(1)}(V_k) | \max_k V_k < m)\mathbb{P}(\max_k V_k < m) \\ &\qquad + \mathbb{P}(\beta^{(2)}(V_i) > \max_{k\neq i} \beta^{(2)}(V_k) | \max_k V_k \geqslant m)\mathbb{P}(\max_k V_k \geqslant m). \end{aligned} \tag{1}$$

So, if we are given the joint distribution F, we can in principle find out the allocation probabilities using the above expression. We also need to identify the optimal bidding strategies in the process.

Now note that conditioned on the event that $\max_k V_k < m$, we need only worry about the first stage English auction and similarly, if $\max_k V_k \geqslant m$, then only the second stage first price auction is of concern.

To deal with the English auction, we make the following important observation.

Observation. If the increments in bids are small compared to the values of participants, then an English auction is approximately like a second price auction.

To see this, suppose that the current bid is b and only two participants (with values v_1 and v_2 such that $v_1 > v_2 \geqslant b$) are willing to pay. Then if the auctioneer increases the bid to $b + \epsilon$ where ϵ is very small compared to v_2 and $b + \epsilon > v_2$, then the second participant walks away, and the first participant wins the auction at price $b + \epsilon \approx v_2$ (note that $1 < \frac{b+\epsilon}{v_2} \leqslant 1 + \frac{\epsilon}{v_2}$). Finally, in English auction, the best strategy for a participant is to hold on until the current price exceeds his/her value. So, at the end, everyone "bids" truthfully. So we will take $\beta^{(1)}(v) = v$.

Therefore, the first term in (1) boils down to

$$\mathbb{P}(V_i > \max_{k\neq i} V_k | \max_k V_k < m)\mathbb{P}(\max_k V_k < m) \tag{2}$$

To handle the second term, we need to know F, because $\beta^{(2)}$ depends on F. In what follows we will assume that $0 < m < 1$ and $V_1, \ldots, V_n \overset{i.i.d.}{\sim}$ Uniform$(0, 1)$. We will use the following result:

Result. If $V_1, \ldots, V_n \overset{i.i.d.}{\sim}$ Uniform$(0, 1)$, then given that $\max_k V_k < m$, $V_1, \ldots, V_n \overset{i.i.d.}{\sim}$ Uniform$(0, m)$.

Proof. Note that for $v_1, \ldots, v_n \in (0, m)$,

$$\begin{aligned}\mathbb{P}(V_1 \leqslant v_1, \cdots, V_n \leqslant v_n | \max_k V_k < m) &= \mathbb{P}(V_1 \leqslant v_1, \cdots, V_n \leqslant v_n)/\mathbb{P}(\max_k V_k < m) \\ &= \frac{v_1}{m} \cdots \frac{v_n}{m}.\end{aligned}$$

□

Therefore,

$$\mathbb{P}(V_i > \max_{k \neq i} V_k | \max_k V_k < m) = \mathbb{P}(Z_i > \max_{k \neq i} Z_k),$$

where $Z_1, \ldots, Z_n \overset{i.i.d.}{\sim}$Uniform$(0, m)$.

The expected revenue corresponding to the English auction is

$$\mathbb{E}(V_{(n-1)} | \max_k V_k < m) = \mathbb{E}(Z_{(n-1)}),$$

where the subscript $(n-1)$ denotes the $(n-1)$-th order statistic, i.e. the second highest number amongst the n numbers of interest. This expectation can be calculated easily.

Unfortunately, given $\max_k V_k \geqslant m$, the joint distribution of $V_1, \ldots, V_n$ is not simple to calculate. So, for now we will skip it and just write down the formula for expected revenue:

$$r = \mathbb{E}(V_{(n-1)} | \max_k V_k < m)\mathbb{P}(\max_k V_k < m) + \mathbb{E}(\max_k \beta^{(2)}(V_k) | \max_k V_k \geqslant m)\mathbb{P}(\max_k V_k \geqslant m).$$

How to find a conditional expectation like $\mathbb{E}(V_{(n-1)} | \max_k V_k < m)$?

We need to know how to calculate the CDF/PDF of an order statistic. Let $X_1, \ldots, X_n \overset{i.i.d.}{\sim} F$ with PDF f. Let $X_{(1)} \leqslant X_{(2)} \leqslant \cdots \leqslant X_{(n)}$ be the n order statistics of $X_1, \ldots, X_n$. Let $F_{(k)}$ and $f_{(k)}$ be respectively the CDf and PDF of $X_{(k)}$. Then

$$\begin{aligned}F_{(k)}(x) = \mathbb{P}(X_{(k)} \leqslant x) &= \mathbb{P}(\text{at least } k \text{ many among the } X_j\text{'s are} \leqslant x) \\ &= \sum_{i=k}^{n} \mathbb{P}(\text{exactly } i \text{ many among the } X_j\text{'s are} \leqslant x) \\ &= \sum_{i=k}^{n} \binom{n}{i} F(x)^i (1 - F(x))^{n-i}.\end{aligned}$$

Therefore,

$$\begin{aligned}
f_{(k)}(x) &= \frac{dF_{(k)}(x)}{dx} \\
&= \frac{d}{dx}\sum_{i=k}^{n}\binom{n}{i}F(x)^i(1-F(x))^{n-i} \\
&= \sum_{i=k}^{n}\binom{n}{i}\frac{d}{dx}[F(x)^i(1-F(x))^{n-i}] \\
&= \sum_{i=k}^{n}\binom{n}{i}[iF(x)^{i-1}f(x)-(n-i)(1-F(x))^{n-i-1}f(x)] \\
&\overset{\text{exercise}}{=} n\binom{n-1}{k-1}f(x)F(x)^{k-1}(1-F(x))^{n-k}
\end{aligned}$$

Example. Suppose $Z_1, \ldots, Z_n \overset{i.i.d.}{\sim}$ Uniform$(0, m)$. Then

$$f_{(k)}(x) = n\binom{n-1}{k-1}\frac{1}{m}\left(\frac{x}{m}\right)^{k-1}\left(1-\frac{x}{m}\right)^{n-k}.$$

In particular, for $k = n-1$, this becomes

$$f_{(n-1)}(x) = n\binom{n-1}{k-1}\frac{1}{m}\left(\frac{x}{m}\right)^{n-2}\left(1-\frac{x}{m}\right).$$

Therefore,

$$\mathbb{E}(Z_{(n-1)}) = \int_0^1 x f_{(n-1)}(x)\,dx.$$

Exercise 1. Calculate the above integral explicitly.

Exercise 2. Using the formula derived above for the density of an order statistic, calculate the expected revenue of a second price auction, when the values are distributed as i.i.d. Exponential(λ).