A scheme for proving Mill's ratio inequalities

Soumendu Sundar Mukherjee

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Abstract

We prove Mill's ratio inequalities via a scheme that provides increasingly sharper bounds and unlike other proofs is easy to remember!

1 Introduction

Mill's ratio for a distribution F with density f is inverse of the hazard rate/survival function, i.e. $\bar{F}(x)/f(x)$, where $\bar{F} = 1 - F$. For the Gaussian distribution a well-known and often used bound for the Mill's ratio (due to Gordon (1941)) is

$$\frac{x}{1+x^2} \cdot \phi(x) < \bar{\Phi}(x) < \frac{1}{x} \cdot \phi(x). \tag{1}$$

Another variant of this inequality is

$$\left(\frac{1}{x} - \frac{1}{x^3}\right) \phi(x) < \bar{\Phi}(x) < \frac{1}{x}\phi(x). \tag{2}$$

In this note we show that inequalities (1) is part of a series of increasingly sharper inequalities. The proof is very easy to remember unlike the standard proofs. To get the upper bound in (1) just integrate the inequality $\bar{\Phi}(t) > 0$ from x to ∞:

$$\begin{aligned} 0 &< \int_x^\infty \bar{\Phi}(t)\, dt \\ &= t\bar{\Phi}(t)|_x^\infty - \int_x^\infty t(-\phi(t))\, dt \\ &= -x\bar{\Phi}(x) + \phi(x). \end{aligned}$$

To get the lower bound, we just integrate the above inequality to get

$$
\begin{aligned}
0 &< \int_x^\infty [-t\bar{\Phi}(t) + \phi(t)]\,dt \\
&= -\frac{t^2}{2}\bar{\Phi}(t)|_x^\infty - \int_x^\infty \frac{t^2}{2}\phi(t)\,dt + \Phi(\bar{x}) \\
&= \frac{x^2}{2}\bar{\Phi}(x) - \frac{t}{2}(-\phi(t))|_x^\infty + \int_x^\infty \frac{1}{2}(-\phi(t))\,dt + \bar{\Phi}(x) \\
&= \frac{x^2}{2}\bar{\Phi}(x) - \frac{x}{2}\phi(x) - \frac{\bar{\Phi}(x)}{2} + \bar{\Phi}(x \\
&= \frac{1}{2}[(1+x^2)\bar{\Phi}(x) - x\phi(x)].
\end{aligned}
$$

You see the structure, right? We can integrate the above inequality to obtain a sharper upper bound

$$\bar{\Phi}(x) < \frac{1}{x} \cdot \frac{2+x^2}{3+x^2} \cdot \phi(x), \tag{3}$$

and so on. This repeated integration trick is quite easy to remember although it is somewhat cumbersome to integrate after a few steps. Below we give a systematic approach.

Proposition 1.1. *Let X be a random variable (with distribution function F) and set $m_k(x) = \mathbb{E}[X^k \mathbf{1}_{X>x}]$. Assume that $\mathbb{E}|X|^N < \infty$. Then for all $0 \le n \le N$ we have*

$$\sum_{k=0}^n \binom{n}{k} m_k(x)(-x)^{n-k} \ge 0.$$

Proof. Note that the left hand side is just $\mathbb{E}(X-x)_+^n \ge 0$:

$$
\begin{aligned}
\mathbb{E}(X-x)_+^n &= \mathbb{E}\sum_{k=0}^n \binom{n}{k} X^k(-x)^{n-k}\mathbf{1}_{X>x} \\
&= \mathbb{E}\sum_{k=0}^n \binom{n}{k} \mathbb{E}[X^k\mathbf{1}_{X>x}](-x)^{n-k} \\
&= \sum_{k=0}^n \binom{n}{k} m_k(x)(-x)^{n-k}.
\end{aligned}
$$

□

To fecilitate calculating $m_k(x)$ we have the following recursion.

Proposition 1.2. *Suppose $E|X|^{k+1} < \infty$. Then*

$$m_{k+1}(x) = x m_k(x) + \int_x^\infty m_k(u)\,du.$$

Proof. Note that

$$\begin{aligned}\int_x^\infty m_k(u)\,du &= \int_x^\infty \mathbb{E}(X^k \mathbf{1}_{X>u})\,du \\ &= \mathbb{E}\left[X^k \int_x^\infty \mathbf{1}_{X>u}\,du\right] \\ &= \mathbb{E}[X^k(X-x)\mathbf{1}_{X>x}] \\ &= m_{k+1}(x) - x m_k(x).\end{aligned}$$

□

Now let's compute using Proposition 1.2 for the Gaussian. If $X \sim N(0,1)$, then

$$\begin{aligned}m_0(x) &= \bar{\Phi}(x) \\ m_1(x) &= \phi(x) \\ m_2(x) &= x\phi(x) + \bar{\Phi}(x) \\ m_3(x) &= x^2\phi(x) + 2\phi(x) \\ m_4(x) &= x^3\phi(x) + 3x\phi(x) + 3\bar{\Phi}(x) \\ &\vdots\end{aligned}$$

Now $n = 1$ in Proposition 1.1 gives

$$m_1(x) - x m_0(x) > 0,$$

i.e.

$$\phi(x) - x\bar{\Phi}(x) > 0,$$

which is the upper bound in (1). Taking $n = 2$ we have

$$m_2(x) - 2x m_1(x) + x^2 m_0(x) > 0,$$

the LHS is

$$x\phi(x) + \bar{\Phi}(x) - 2x\phi(x) + x^2\bar{\Phi}(x) = (1 + x^2)\bar{\Phi}(x) - x\phi(x).$$

So we get the lower bound in (1). Taking $n = 3$ gives (3), while $n = 4$ gives

$$\bar{\Phi}(x) \geq x \cdot \frac{1}{1+x^2} \cdot \frac{5 + 6x + x^2}{3 + 6x + x^2} \cdot \phi(x)$$

and so on.

References

Gordon, R. D. (1941). Values of mills' ratio of area to bounding ordinate and of the normal probability integral for large values of the argument. *The Annals of Mathematical Statistics*, 12(3):364–366.