A NOTE ON FIRST PRICE AUCTION

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Note: If you have comments or questions, feel free to email me at soumendu@berkeley.edu.

The purpose of this note is to derive a formula for Bayes-Nash equilibrium for a first price auction. Suppose there are n bidders with values $V_1, \ldots, V_n \overset{i.i.d.}{\sim} F$, where F is a differentiable CDF with density $f(x)$. Let $\beta : [0, \infty) \to [0, \infty)$ be a strictly increasing bidding strategy and $\beta_1 = \cdots = \beta_n = \beta$ be a Bayes-Nash equilibrium.

The utility to bidder i when he/she bids $\beta(w)$ is

$$\begin{aligned} u(w|v_i) &= (v_i - \beta(w))\mathbb{P}(\beta(w) > \max_{j \neq i} \beta(V_j)) \\ &= (v_i - \beta(w))\mathbb{P}(w > \max_{j \neq i} V_j) \\ &= (v_i - \beta(w))F(w)^{n-1}. \end{aligned}$$

Therefore,

$$\frac{\partial u(w|v_i)}{\partial w} = (v_i - \beta(w))(n-1)F(w)^{n-2}f(w) - \beta'(w)F(w)^{n-1}.$$

Since β is a Bayes-Nash equilibrium, $w \mapsto u(w|v_i)$ is maximized at $w = v_i$ for each i; so we must have

$$\frac{\partial u(w|v_i)}{\partial w}|_{w=v_i} = 0,$$

which means that

$$(v_i - \beta(v_i))(n-1)F(v_i)^{n-2}f(v_i) - \beta'(v_i)F(v_i)^{n-1} = 0,$$

for each i. Writing $v_i = v$ and rearranging we arrive at

$$\beta'(v)F(v)^{n-1} + (n-1)F(v)^{n-2}f(v)\beta(v) = (n-1)vF(v)^{n-2}f(v),$$

which is same as

$$(\beta(v)F(v)^{n-1})' = (n-1)vF(v)^{n-2}f(v).$$

Integrating and using the fact that $F(0) = 0$ we get

$$\beta(v)F(v)^{n-1} = (n-1)\int_0^v tF(t)^{n-2}f(t)\,dt,$$

which finally gives us

$$\boxed{\beta(v) = \frac{(n-1)\int_0^v tF(t)^{n-2}f(t)\,dt}{F(v)^{n-1}}}. \tag{1}$$

Example. Suppose F is the CDF of Uniform$(0, 1)$. Then using the formula above we get

$$\begin{aligned}\beta(v) &= \frac{(n-1)\int_0^v t^{n-1}\,dt}{v^{n-1}} \\ &= \frac{n-1}{n}v.\end{aligned}$$

Recall that we worked out the cases $n = 2, 3$ previously. This reveals the following interesting fact. For large n,

$$\beta(v) = \frac{n-1}{n}v \approx v.$$

This implies that for large n first price auction is *approximately truthful*. Thus for large n, first and second price auctions are approximately the same. As an exercise you can try to prove that for any n the allocation probabilities for first price and second price auctions are the same, and then by revenue equivalence theorem conclude that the expected revenues are also the same.

Exercise. Suppose F is the CDF of Exponential(λ). Find out β using Formula 1.