1. Suppose \((\mathcal{M}_1, \rho_1), (\mathcal{M}_2, \rho_2)\) are two metric spaces and let \(\mathcal{M} = (\mathcal{M}_1 \times \mathcal{M}_2, \rho_1 \vee \rho_2)\) be their product. Show that

(a) \(\mathcal{M}\) is separable if and only if \(\mathcal{M}_1\) and \(\mathcal{M}_2\) are so.

(b) Let \(P, P_n, n \geq 1\) be probability measures on \((\mathcal{M}, \mathcal{B}(\mathcal{M}))\). Assuming \(\mathcal{M}\) to be separable, show that \(P_n \xrightarrow{w} P\) if and only if \(P_n(A_1 \times A_2) \rightarrow P(A_1 \times A_2)\) for all \(P \circ \pi^{-1}_1\) continuity sets \(A_1\) and \(P \circ \pi^{-1}_2\) continuity sets \(A_2\), where \(\pi_i\)'s are the co-ordinate projections.

(c) Let \(\mathcal{M}_i = [0, 1]\) equipped with the Euclidean metric. Let \(P_1\) be the uniform distribution on \(\mathcal{M}\), and \(P_2\) be the uniform distribution on the diagonal \(\{(x, x) \mid x \in [0, 1]\}\). Show that \(P_1\) and \(P_2\) have identical marginals.

(d) Using (c) or otherwise, show, in the context of (b), that \(P_n \circ \pi^{-1}_1 \xrightarrow{w} P \circ \pi^{-1}_1\) and \(P_n \circ \pi^{-1}_2 \xrightarrow{w} P \circ \pi^{-1}_2\) together do not imply that \(P_n \xrightarrow{w} P\).

\[4 + 10 + 3 + 3\]

2. (a) Show that \(\lambda : [0, 1] \rightarrow [0, 1]\) is a homeomorphism with \(\lambda(0) = 0, \lambda(1) = 1\) if and only if \(\lambda\) is a continuous strictly increasing surjection on \([0, 1]\). Show that such \(\lambda\)'s form a group with respect to function composition.

(b) Define the Skorohod metric \(d\) on \(D[0, 1]\).

(c) Let \(\phi_n(t) = 1_{[\frac{i}{n}, \frac{i+1}{n}]}(t)\) and \(\phi(t) = 1_{[0, 1]}(t)\). Show that \(d(\phi_n, \phi) \rightarrow 0\).

(d) Show that if \(x \in C[0, 1]\) and \(x_n \in D[0, 1], n \geq 1\), then \(d(x_n, x) \rightarrow 0 \implies \|x_n - x\|_{\infty} \rightarrow 0\).

(e) Show that if \(x \in D[0, 1]\) and \(x_n \in C[0, 1], n \geq 1\), then \(d(x_n, x) \rightarrow 0 \implies \|x_n - x\|_{\infty} \rightarrow 0\).

\[(2 + 3) + 2 + 5 + 4 + 4\]

3. Let \(\xi_i \overset{i.i.d.}{\sim} F, i = 1, \ldots, n\). Consider the empirical process

\[X_n(t) = \sqrt{n}(\frac{1}{n} \sum_{1 \leq i \leq n} 1_{\{\xi_i \leq t\}} - F(t)).\]

Derive the limit distribution of \((X_n(t_1), \ldots, X_n(t_k))\), where \(t_i \in (0, 1)\). State Donsker’s theorem for \(X_n\).

Now suppose that \(F\) is the CDF of the uniform distribution on \([0, 1]\). Then \(X_n\) is the uniform empirical process. State with proper justification if \(X_n\) is measurable with respect to
(i) the Borel σ-field on \((D[0, 1], \| \cdot \|_\infty)\),
(ii) the ball σ-field on \((D[0, 1], \| \cdot \|_\infty)\), and
(iii) the Borel σ-field in the Skorohod topology on \(D[0, 1]\).

4. (a) Suppose \(\mathcal{F}, \mathcal{F}_1, \mathcal{F}_2\) are \(\mathbb{P}\)-Glivenko-Cantelli classes. State with proper justification which of the following are \(\mathbb{P}\)-Glivenko-Cantelli classes:

(i) \(\mathcal{F}_1 \cup \mathcal{F}_2\),
(ii) \(\mathcal{F}_1 \cap \mathcal{F}_2\),
(iii) \(\{a_1 f_1 + a_2 f_2 | f_i \in \mathcal{F}_i, |a_i| \leq 1\}\),
(iv) \(\{f | \exists\) a sequence \((f_m)_{m \geq 1}\) in \(\mathcal{F}\) such that \(f_m \to f\) pointwise as well as in \(L_1(\mathbb{P})\}\).

(b) Suppose \(X_i, i = 1, \ldots, n\) are independent random variables defined on some probability space \((\Omega, \mathcal{A}, \mathbb{P})\) and taking values in a Hilbert space \((\mathcal{H}, \langle \cdot, \cdot \rangle_\mathcal{H})\). Let \(\| \cdot \|_\mathcal{H}\) be the corresponding norm. Suppose that \(\|X_i\|_\mathcal{H} \leq b_i\) almost surely, for constants \(b_i > 0, i = 1, \ldots, n\). Consider the random variable \(S_n = \| \sum_{i=1}^n X_i \|_\mathcal{H}\). Show that

\[
\mathbb{P}(\|S_n - \mathbb{E}S_n\| \geq n\delta) \leq 2 \exp \left( - \frac{n^2 \delta^2}{8 \sum_{i=1}^n b_i^2} \right).
\]

5. (a) Define covering and bracketing numbers. Give an example where these are equivalent (up to constant factors).

(b) State and prove the classical Glivenko-Cantelli theorem using \(L_1\)-bracketing numbers.

(c) The \(\delta\)-packing number \(M(\delta; \mathcal{M}, \rho)\) of a (pseudo-)metric space \((\mathcal{M}, \rho)\) is the maximum cardinality of a set of points in \(\mathcal{M}\) that are more than \(\delta\) apart in \(\rho\). If \(N(\delta; \mathcal{M}, \rho)\) denotes the \(\delta\)-covering number of \((\mathcal{M}, \rho)\), show that

\[
M(2\delta; \mathcal{M}, \rho) \leq N(\delta; \mathcal{M}, \rho) \leq M(\delta; \mathcal{M}, \rho).
\]