Indian Statistical Institute  
Mid-semestral Examination  
February 25, 2019  

Weak Convergence and Empirical Processes, M2

Total points: 20  
Time: 1\frac{1}{2} hours

Note: This is a closed notes/closed book examination. Do any four of the following problems. Notations, if not explicitly explained, are to be interpreted as defined in class.

1. Closure of open balls.  
Let $(\mathcal{M}, \rho)$ be a metric space, $x \in \mathcal{M}, \epsilon > 0$. Show that $\overline{B(x; \epsilon)} \neq B[x; \epsilon]$ in general. If, however, $\mathcal{M}$ is a normed vector space, with $\rho$ coming from the norm, then show that $\overline{B(x; \epsilon)} = B[x; \epsilon]$.

2. Tightness of a Gaussian family.  
Let $A \subset \mathbb{R} \times (0, \infty)$. Set $\Pi_A = \{P_{\mu, \sigma^2} \mid (\mu, \sigma^2) \in A\}$, where $P_{\mu, \sigma^2}$ is a $\mathcal{N}(\mu, \sigma^2)$ probability measure. Show that $\Pi_A$ is tight if and only if $A$ is bounded.

3. Weak convergence in countable and discrete spaces.  
Suppose that $\mathcal{M}$ is countable and discrete, and let $\mathbb{P}_n, \mathbb{P}$ be Borel probability measures on $\mathcal{M}$. Show that $\mathbb{P}_n \xrightarrow{w} \mathbb{P}$ if and only if $\mathbb{P}_n\{x\} \to \mathbb{P}\{x\}$ for all $x \in \mathcal{M}$. Show also that, in this case, $d_{TV}(\mathbb{P}_n, \mathbb{P}) := \sup_{A \in \mathcal{B}(\mathcal{M})} |\mathbb{P}_n(A) - \mathbb{P}(A)| \to 0$.

4. Some Brownian computations.  
(a) Let $X, Y, Z$ be random variables defined on a common probability space. If $(X, Y) \perp Z$, then show that $\mathbb{E}[X \mid Y, Z] = \mathbb{E}[X \mid Y]$ a.s.
(b) Using the above result, or otherwise, compute (i) $\mathbb{E}[B(s) \mid B(r), B(t)]$, (ii) $\text{Var}[B(s) \mid B(r), B(t)]$, where $0 \leq r < s < t \leq 1$.

5. The number of levels reached by a simple symmetric random walk.  
Consider a simple symmetric random walk $(S_k)_{k \geq 0}$, and let $L_n$ denote the number of levels reached by the random walk by time $n$. Thus $L_0 = 1, L_1 = 2, L_2 \in \{2, 3\}$, and so on. Show that $\frac{L_n}{\sqrt{n}}$ converges weakly and give an expression for its limit CDF.